DISCUSSION

OF A

GEOMETRICAL PROBLEM:

WITH BIBLIOGRAPHICAL NOTES.

BY

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To Mr George Eastwood
with the author's compliments.

DISCUSSION OF A GEOMETRICAL PROBLEM, WITH BIBLIOGRAPHICAL NOTES. BY MARCUS BAKER, U. S. COAST SURVEY, WASHINGTON, D. C.

The problem here discussed, and of which several solutions are given, is the following :—

In a right-angled triangle there are given the bisectors of the acute angles: required to determine the triangle.

This problem, like most problems in triangles in which the bisectors of the angles enter as a part of the data, cannot be solved by the elements of geometry, *i. e.* by the use of the circle and right line only. We shall give, first, trigonometrical solutions; second, algebraical solutions; third, constructions; and fourth, bibliographical notes.

FIRST SOLUTION.

Let α and β be the bisectors of the angles A and B respectively: then we have

$AB \sin A = \beta \cos (45° - \frac{1}{2} A)$ and $AB \cos A = \alpha \cos \frac{1}{2} A$;

whence by dividing, remembering that

$$\frac{\cos (45° - \frac{1}{2} A)}{\cos \frac{1}{2} A} = \frac{1 + \tan \frac{1}{2} A}{\sqrt{2}}.$$

$$\tan A = \frac{\beta}{\alpha \sqrt{2}} (1 + \tan \frac{1}{2} A) : . . . \qquad (1)$$

and since

$$\tan A = \frac{2 \tan \frac{1}{2} A}{1 - \tan^2 \frac{1}{2} A},$$

we obtain by reduction

$$\tan^3 \tfrac{1}{2} A + \tan^2 \tfrac{1}{2} A + \left(\frac{\alpha}{\beta} \sqrt{8} - 1\right) \tan \tfrac{1}{2} A - 1 = 0 \; . \; . \; . \qquad (2)$$

from which equation we may find $\tan \frac{1}{2} A$.

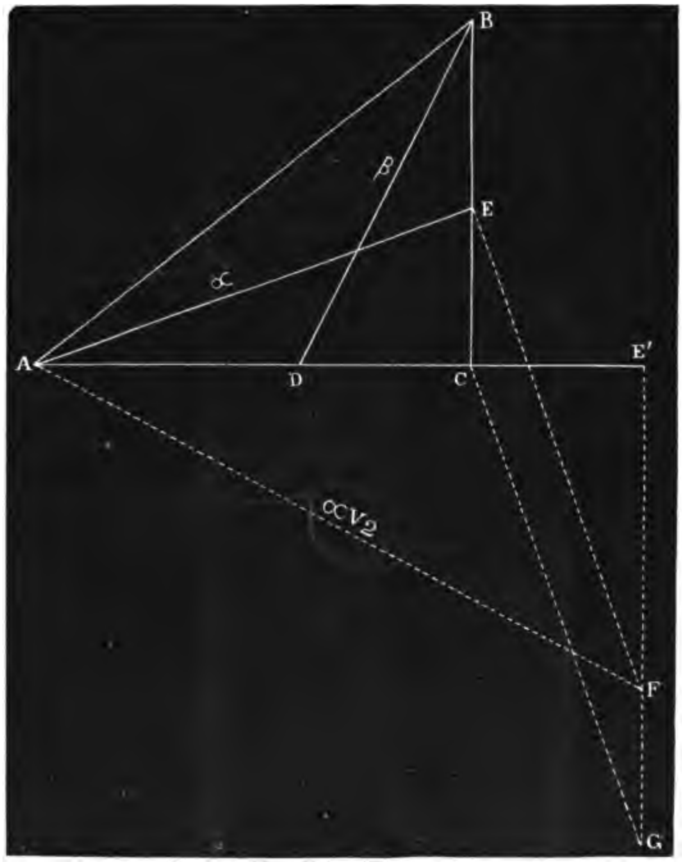
We may, however, obtain Eq. (1) directly from a construction as follows :—

Prolong A C to E′ making C E′ = C E, and from E′ draw E′ G perpendicular to A E′: from E draw E F perpendicular to A E, meeting E′ G in F; and from C draw C G parallel to E F. Now the triangle C E′ G is equal to the triangle A C E; hence

C G = E A, and also E F = E A: hence A E F is an isosceles right-angled triangle and A F $= \alpha\sqrt{2}$. Also B D C and A F E′ are similar triangles: whence

$$BC : AE' :: \beta : \alpha\sqrt{2}.$$

Fig. 1.



Now when A C = radius, or 1, B C = tan A and A E′ = 1 + tan $\frac{1}{2}$ A: whence

$$\tan A = \frac{\beta}{\alpha\sqrt{2}}\left(1 + \tan\tfrac{1}{2}A\right)$$

as before.

In this solution we have selected as our unknown quantity

$\tan \frac{1}{2} A$. We might obviously have selected any other trigonometrical function, but this seems to lead to as simple a result as any.

If we make $\sin \frac{1}{2} A$ our unknown quantity our equation will be

$$\left\{ \begin{array}{c} 4\left\{(1-\frac{\alpha}{\beta}\sqrt{2})^2+1\right\}\sin^6\frac{1}{2}A-4\left\{\left(1-\frac{\alpha}{\beta}\sqrt{2}\right)\left(1-\frac{\alpha}{\beta}2\sqrt{2}\right)+2\right\} \\ \sin^4\frac{1}{2}A-4\left\{1+\frac{\alpha}{\beta}\sqrt{2}-\frac{2\alpha^2}{\beta^2}\right\}\sin^2\frac{1}{2}A-1=0, \end{array} \right\}$$

and if we make $\sec \frac{1}{2} A$ the unknown quantity our equation will be

$$\left\{ \begin{array}{l} \sec^6\frac{1}{2}A-2\left(3-\frac{\alpha}{\beta}\sqrt{8}\right)\sec^4\frac{1}{2}A+2\left(6-\frac{3\alpha}{\beta}\sqrt{8}+\right. \\ \left.\frac{4\alpha^2}{\beta^2}\right)\sec^2\frac{1}{2}A-4\left(1-\frac{\alpha}{\beta}\sqrt{8}+\frac{2\alpha^2}{\beta^2}\right)=0; \end{array} \right\}$$

whence it appears that the simplest equation is the one first obtained in which the tangent is made the unknown quantity.

Example.—Suppose $\alpha=40$ and $\beta=50$. Then our equation becomes

$$\tan^3\tfrac{1}{2}A+\tan^2\tfrac{1}{2}A+\left(\tfrac{4}{5}\sqrt{8}-1\right)\tan\tfrac{1}{2}A-1=0;$$

whence by Horner's method

$$\tan\tfrac{1}{2}A = 0.49788,\quad 15817,\quad 54736.$$

Whence $\quad A = 37^\circ\ 03'\ 51''.33$

$\quad\quad\quad\quad B = 52^\circ\ 56'\ 08''.67,$

and the sides of the triangle are

$$a = 35.807377$$
$$b = 47.407275$$
$$c = 59.41058.$$

SECOND SOLUTION.

Let a, b, and c be the sides of the triangle opposite A, B, and C respectively, and α and β as before; then we have (Fig. 1)

$$\frac{b}{\alpha}=\cos\tfrac{1}{2}A;\ \text{whence}\ \frac{2b^2}{\alpha^2}=2\cos^2\tfrac{1}{2}A=1+\cos A=1+\frac{b}{c};$$

therefore

$$\frac{2b}{\alpha^2}=\frac{1}{b}+\frac{1}{c},$$

and similarly

$$\frac{2a}{\beta^2}=\frac{1}{a}+\frac{1}{c};$$

whence

$$\frac{2b}{\alpha^2}-\frac{1}{b}=\frac{2a}{\beta^2}-\frac{1}{a}=\frac{1}{c}. \qquad (3)$$

Again

$$\frac{a}{\beta}=\cos\tfrac{1}{2}\mathrm{B}=\cos(45^\circ-\tfrac{1}{2}\mathrm{A})=\frac{\sin\frac{1}{2}\mathrm{A}+\cos\frac{1}{2}\mathrm{A}}{\sqrt{2}};$$

whence

$$\sin\tfrac{1}{2}\mathrm{A}=\frac{a\sqrt{2}}{\beta}-\cos\tfrac{1}{2}\mathrm{A}=\frac{a\sqrt{2}}{\beta}-\frac{b}{\alpha};$$

and since $\sin^2\frac{1}{2}\mathrm{A}+\cos^2\frac{1}{2}\mathrm{A}=1$,

2

$$\left(\frac{a\sqrt{2}}{\beta}-\frac{b}{\alpha}\right)^2+\frac{b^2}{\cancel{\alpha}}=1,$$

or

$$\frac{2a^2}{\beta^2}-\frac{2\sqrt{2}\,a\,b}{\alpha\,\beta}+\frac{2b^2}{\alpha^2}=1. \qquad (4)$$

If now we eliminate b between Eqs. (3) and (4) we have an equation from which a may be found.

From (4) we find, $b=\frac{\alpha}{\beta\sqrt{2}}\left\{a\pm\sqrt{\beta^2-a^2}\right\}$ which substitut-

in ed ~~m~~ (3) gives after some reduction

$$\frac{2a^2-\beta^2}{a}=\frac{\pm 2ma\sqrt{\beta^2-a^2}}{a\pm\sqrt{\beta^2-a^2}}$$

where $m=\frac{\beta}{\alpha}\sqrt{2}$. This equation finally reduces to

$$(\alpha^2-\alpha\beta\sqrt{2}+\beta^2)\,a^6-(3\alpha^2-3\sqrt{2}\,\alpha\beta+2\beta^2)\frac{\beta^2}{2}a^4+(3\alpha^2-2\sqrt{2}\,\alpha\beta)\frac{\beta^4}{4}a^2-\frac{\alpha^2\beta^6}{8}=0. \qquad (5)$$

THIRD SOLUTION.

Revolve the triangles B O E and D O A about B O and A O respectively so that E falls upon E′ and D upon D′, then

$$\mathrm{EOB}=\mathrm{E'OB}=\mathrm{E'OD'}=\mathrm{D'OA}=\mathrm{AOD}=45^\circ,$$

and consequently B O D′ and A O E′ are right-angled triangles: hence

$$\frac{\mathrm{OE}}{\mathrm{OA}}=\tan\tfrac{1}{2}\mathrm{A},\ \text{ or }\ \frac{\alpha}{\mathrm{OA}}=1+\tan\tfrac{1}{2}\mathrm{A};$$

whence $\alpha=\mathrm{OA}\,(1+\tan\frac{1}{2}\mathrm{A})$, (6)

and similarly $\beta=\mathrm{OB}\,(1+\tan\frac{1}{2}\mathrm{B})$. (7)

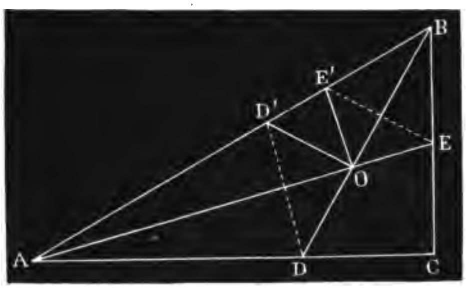
Again, $O A \sin \frac{1}{2} A = \chi$; whence from (6)

$$\frac{\alpha}{\chi} = \frac{1 + \tan \frac{1}{2} A}{\sin \frac{1}{2} A},$$

or $\dfrac{\alpha}{\chi} = \dfrac{1}{\sin \frac{1}{2} A} + \dfrac{1}{\cos \frac{1}{2} A}$, and similarly

$$\frac{\beta}{\chi} = \frac{1}{\sin \frac{1}{2} B} + \frac{1}{\cos \frac{1}{2} B}.$$

Fig. 2.



Now $\sin \frac{1}{2} B = \sin (45^\circ - \frac{1}{2} A) = \dfrac{\cos \frac{1}{2} A - \sin \frac{1}{2} A}{\sqrt{2}}$ and

$$\cos \frac{1}{2} B = \cos (45^\circ - \tfrac{1}{2} A) = \frac{\cos \frac{1}{2} A + \sin \frac{1}{2} A}{\sqrt{2}};$$

whence

$$\frac{\beta}{\chi \sqrt{2}} = \frac{1}{\cos \frac{1}{2} A - \sin \frac{1}{2} A} + \frac{1}{\cos \frac{1}{2} A + \sin A} = \frac{2 \cos \frac{1}{2} A}{2 \cos^2 \frac{1}{2} A - 1}.$$

from which we find

$$\cos \frac{1}{2} A = \frac{1}{\sqrt{2}} \left\{ \frac{\chi}{\beta} \pm \sqrt{1 + \frac{\chi^2}{\beta^2}} \right\},$$

and similarly

$$\cos \frac{1}{2} B = \frac{1}{\sqrt{2}} \left\{ \frac{\chi}{\alpha} \pm \sqrt{1 + \frac{\chi^2}{\alpha^2}} \right\}.$$

Since $\cos \frac{1}{2} B = \dfrac{\cos \frac{1}{2} A + \sin \frac{1}{2} A}{\sqrt{2}}$,

$$\sin \frac{1}{2} A = \left\{ \frac{\chi}{\alpha} \pm \sqrt{1 + \frac{\chi^2}{\alpha^2}} \right\} - \frac{1}{\sqrt{2}} \left\{ \frac{\chi}{\beta} \pm \sqrt{1 + \frac{\chi^2}{\beta^2}} \right\};$$

whence

$$\left\{ \frac{\chi}{\alpha} \pm \sqrt{1 + \frac{\chi^2}{\alpha^2}} \right\}^2 - \sqrt{2} \left\{ \frac{\chi}{\alpha} \pm \sqrt{1 + \frac{\chi^2}{\alpha^2}} \right\} \left\{ \frac{\chi}{\alpha} \pm \sqrt{1 + \frac{\chi^2}{\beta^2}} \right\}$$
$$+ \left\{ \frac{\chi}{\alpha} \pm \sqrt{1 + \frac{\chi^2}{\beta^2}} \right\} = 1. \qquad (8)$$

This equation involves only x, the radius of the inscribed circle and the given bisectors of the angles α and β: hence we may determine x from it. Eq. (8) becomes after a somewhat laborious reduction

$$64(\alpha^2 - \alpha\beta\sqrt{2} + \beta^2)x^6 + 8\sqrt{2}\,\alpha\beta(4\alpha^2 - 3\sqrt{2}\,\alpha\beta + 4\beta^2)x^4 + \alpha^2\beta^2(2\alpha^2 - \alpha\beta\sqrt{2} + 2\beta^2)x^2 - \alpha^4\beta^4 = 0. \qquad (9)$$

These three solutions just given all involve trigonometrical relations and are therefore properly classed as trigonometric solutions. They may all, however, be made independently of trigonometry. In the following we shall give the algebraical solutions corresponding to the first and second trigonometrical solutions together with a third and entirely independent solution.

ALGEBRAICAL SOLUTION.

From Fig. 2 we have

$$c : \mathrm{AD} :: a : \mathrm{DC} :: \mathrm{OB} : \mathrm{OD} :: 1 : n$$
$$c : \mathrm{BE} :: b : \mathrm{EC} :: \mathrm{OA} : \mathrm{OE} :: 1 : m;$$

from which

$$\mathrm{OB} = \frac{\beta}{1+n},\ \mathrm{OD} = \frac{\beta n}{1+n},\ \mathrm{OA} = \frac{\alpha}{1+m},\ \mathrm{OE} = \frac{\alpha m}{1+m},$$
$$\mathrm{AD} = cn,\ \mathrm{CD} = an,\ \mathrm{BE} = cm,\ \mathrm{CE} = bm.$$

Now

$$c.\,\mathrm{AD} = \mathrm{OA}^2 + \mathrm{OB.OD} \text{ or } c^2 n = \left(\frac{\alpha}{1+m}\right)^2 + n\left(\frac{\beta}{1+n}\right)^2,$$

$$c.\,\mathrm{BE} = \mathrm{OB}^2 + \mathrm{OA.OE} \text{ or } c^2 m = \left(\frac{\beta}{1+n}\right)^2 + m\left(\frac{\alpha}{1+m}\right)^2;$$

whence

$$m\left(\frac{\alpha}{1+m}\right)^2 + mn\left(\frac{\beta}{1+n}\right)^2 = n\left(\frac{\beta}{1+n}\right)^2 + m\gamma\left(\frac{\alpha}{1+m}\right)^2,$$

or

$$\frac{m(1-n)}{(1+m)^2}\alpha^2 = \frac{n(1-m)}{(1+n)^2}\beta^2. \qquad (10)$$

Again $b = \mathrm{AD} + \mathrm{CD} = n(c+a) \therefore a^2 + n^2(c+a)^2 = c^2$ and $a = \mathrm{CE} + \mathrm{BE} = bm + cm = mn(c+a) + cm$

$$= cm(1+n) + amn \therefore \frac{c}{a} = \frac{1-mn}{m(1+n)}.$$

Equating these two expressions

$$\frac{1-mn}{m} = \frac{1+n^2}{1-n} \therefore m = \frac{1-n}{1+n} \text{ and } n = \frac{1-m}{1+m};$$

substituting in (10) we find after reducing

$$\left.\begin{array}{l} n^3 + n^2 + \left(\sqrt{8}\,\frac{\beta}{\alpha} - 1\right)n - 1 = 0 \\ m^3 + m^2 + \left(\sqrt{8}\,\frac{\alpha}{\beta} - 1\right)m - 1 = 0. \end{array}\right\} \qquad (11)$$

It is to be noted that $n = \frac{\text{D C}}{a} = \tan \frac{1}{2}\,\text{B}$ and $m = \frac{\text{E C}}{b} = \tan \frac{1}{2}\,\text{A}$, and therefore Eq. (11) corresponds to Eq. (2).

FIFTH SOLUTION.

The fundamental relations between the sides and bisectors are

$$\alpha^2 = \frac{bc\,(a+b+c)\,(-a+b+c)}{(b+c)^2} = (b^2 + 2\,bc + c^2 - a^2)\,\frac{bc}{(b+c)^2}$$

$$\beta^2 = \frac{ac\,(a+b+c)\,(a-b+c)}{(a+c)^2} = (a^2 + 2\,ac + c^2 - b^2)\,\frac{ac}{(a+c)^2}.$$

And since $a^2 + b^2 = c^2$

$$\alpha^2 = 2\,b^2\frac{c}{b+c} \text{ or } \frac{2\,b^2}{\alpha^2} = \frac{b+c}{c} = 1 + \frac{b}{c},$$

$$\beta^2 = 2\,a^2\,\frac{c}{a+c} \text{ or } \frac{2\,a^2}{\beta^2} = \frac{a+c}{c} = 1 + \frac{a}{c}.$$

Whence

$$\frac{2\,b}{\alpha^2} - \frac{1}{b} = \frac{2\,a}{\beta^2} - \frac{1}{a} = \frac{1}{c} \qquad (3)$$

as in the second solution, where this relation was obtained trigonometrically. Again

$$\frac{8\,a^2b^2}{\alpha^2\beta^2} = 2\left(1 + \frac{a}{c}\right)\left(1 + \frac{b}{c}\right) = 2\left\{1 + \frac{a+b}{c} + \frac{ab}{c^2}\right\} =$$

$$2\left\{\frac{a^2 + ab + b^2}{a^2} + \frac{a+b}{c}\right\} = \frac{a^2 + 2\,ab + b^2}{c^2} + 2\,\frac{a+b}{c} + 1$$

$$\therefore\ \frac{2\sqrt{2\,ab}}{\alpha\beta} = \frac{a+b}{c} + 1.$$

Again by adding

$$\frac{2\,a^2}{\beta^2} + \frac{2\,b^2}{\alpha^2} = \frac{a+b}{c} + 2.$$

Whence

$$\frac{2\,a^2}{\beta^2} - \frac{2\sqrt{2\,ab}}{\alpha\beta} + \frac{2b^2}{\alpha^2} = 1 \qquad (4)$$

as previously obtained trigonometrically. The solution is now completed as in the second solution.

SIXTH SOLUTION.

Let $OE = OE' = x$ (Fig. 2), $OD = OD' = y$, $AE = a$, and $BD = \beta$; the angles marked with a dot are each equal to 45°, and therefore $EE' = x\sqrt{2}$, and $DD' = y\sqrt{2}$.

From similar triangles $BO : BD = OE' : DD'$, or $\beta - y : \beta :: x : y\sqrt{2}$. Whence

$$(\beta - y)\, y\sqrt{2} = \beta x. \tag{12}$$

$$(a - x)\, x\sqrt{2} = a y. \tag{13}$$

From (13)

$$y = \frac{\sqrt{2}}{a} x (a - x), \text{ and substituting in (12)}$$

$$\beta - \frac{\sqrt{2}}{a} x (a - x) = \frac{a\beta}{2(a - x)}.$$

Which reduces to

$$(a - x)^2 x = \frac{a\beta}{2\sqrt{2}} (a - 2x).$$

Expanding, rearranging, etc., this reduces to

$$x^3 - 2ax^2 + a\left(\frac{\beta}{\sqrt{2}} + a\right) x - \frac{a\beta}{2\sqrt{2}} = 0 \tag{14}$$

CONSTRUCTIONS.

First Construction.—The equations obtained in the sixth solution point to a simple construction of the problem, as follows:—

Equations (12) and (13) may be written as follows:—

$$x^2 - ax + \frac{a}{\sqrt{2}} y = 0. \tag{15}$$

$$y^2 - \beta y + \frac{\beta}{\sqrt{2}} x = 0. \tag{16}$$

And each of these equations is the equation of a parabola. If these two parabolas be constructed, their intersection will determine x and y. The position and size of the parabola will readily appear by transforming co-ordinates. In equation (15) let

$$x = x' + \frac{a}{2} \text{ and}$$

$$y = y' + \frac{a}{2\sqrt{2}}, \text{ then}$$

$$x'^2 = -\frac{a}{\sqrt{2}} y';$$

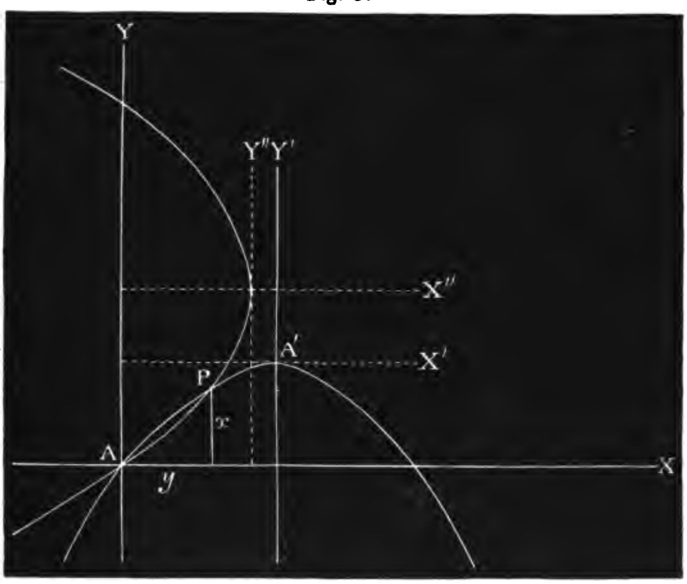
and in equation (16) let

$$y = y'' + \frac{\beta}{2} \text{ and } x = x'' + \frac{\beta}{2\sqrt{2}}, \text{ then}$$

$$y''^2 = -\frac{\beta}{\sqrt{2}} x''.$$

The following construction results immediately from the above. With reference to a set of co-ordinates X A Y construct a new set X′ A′ Y′ such that $x - x' = \frac{2}{\alpha}$ and $y - y' = \frac{\alpha}{2\sqrt{2}}$, and another set X″ A″ Y″ such that $x - x'' = \frac{\beta}{2\sqrt{2}}$ and $y - y'' = \frac{\beta}{2}$. With the first new set construct the parabola $x' = -\frac{\alpha}{\sqrt{2}} y$, and with the second new set construct the parabola $y''^2 = -\frac{\beta}{\sqrt{2}} x''$;

Fig. 3.



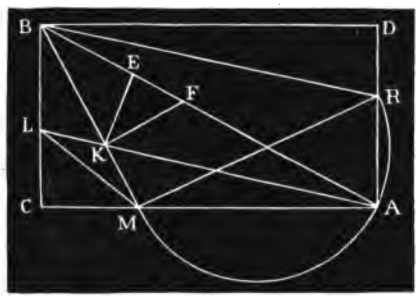
their intersections will determine the segments x and y, *i. e.*, O E and O D of Fig. 2. The construction is shown in Fig. 3.

Second Construction.—Take a rectangle A C B D, Fig. 4, and let A L, B M, the bisectors of A and B, intersect in K; then A K B = 135°. Through B draw B R parallel to A L to meet

A D in R; then B R $=$ A L. Hence from data the triangle B M R is known.

It is well known that $2\triangle BMR + CM \cdot DR =$ rect. $AB = 2\triangle BMR + 2\triangle CLM$. (17)

Fig. 4.



Take $BE = BL$, $AF = AM$; then $\triangle BKE = \triangle BKL$, $\triangle AKF = \triangle AKM$, and $\triangle FKE = \triangle LKM$, because the angles L K M, F K E, are supplementary; therefore $\square AMLB = 2\triangle AKB$; hence by (17) $\triangle AKB = \frac{1}{2}\triangle BMR$.

Construction.—Make a triangle B M R, having its sides B M, B R, equal to the given bisectors, and the angle M B R equal to half a right angle. On M R draw a semicircle, and construct a hyperbola having B M, B R, for asymptotes, and such that the rectangle under the ordinate and abscissa (parallel to the asymptotes) is half the rectangle under the given bisectors. Let this hyperbola cut the semicircle in A; join A B and produce A K parallel to B R, so that $AL = BR$; and produce B L, A M, to meet in C. Then A B C will be the triangle required.

BIBLIOGRAPHICAL NOTES AND ACKNOWLEDGMENTS.

This problem was proposed in the Ladies' Diary for 1797, by Alex. Rowe, and the following year two solutions of it were given; one by William Burdon and the other by J. Hartley. Our sixth solution is taken from Mr. Burdon, as published in Leybourn (Thomas). The Mathematical Questions proposed in the Ladies' Diary, etc., 8vo., London, 1817, vol. iii. 328.

Mr. Hartley's solution is trigonometrical, the unknown quantity being $\tan \frac{1}{2} A$, and his final equation corresponds to equation

(2), but the mode of obtaining it is not so elegant as that employed in our first solution.

The problem is proposed as an exercise in Bonnycastle (John). An Introduction in Algebra, etc., revised and enlarged, by James Ryan, 4th edition, 12mo., New York, 1829, p. 310. In the key to the second edition, New York, 1822, pp. 250–251, is a solution essentially the same as the first one given here.

The problem extended to any triangle was proposed by the writer in the Analyst, vol. iii., No. 5, Sept. 1876, p. 163, and solved in the next number, pp. 188–189, by Prof. J. Scheffer. It was also solved by Henry Gunder, William Hoover, and the writer.

The problem not extended was proposed in the Educational Times of January 1, 1879, p. 22, question 5866, by Mr. N. H. Capel; and in the following number proposed by the editor for construction, question 5885. In the May number, p. 150, a construction by Mr. R. Tucker was given, which we have here incorporated verbatim as our second construction.

For the fourth solution I am indebted to my classmate, Prof. W. W. Beman, of the University of Michigan.